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DETERMINATION OF TEMPERATURE AT THE OUTER BOUNDARY OF A BODY

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We study the problem of finding the temperature on the outer boundary of a cylindrical domain under given conditions on the inner boundary. We establish the existence and uniqueness of a solution. Bibliography: 13 *titles.*

1 Introduction

1.1. In this paper, we establish the solvability of the boundary value problem associated with the following physical model. Suppose that a rod radiating heat is placed in a cylindrical inhomogeneous domain $\Omega \times \mathbb{R}$. The temperature on the rod surface and the heat flow through the rod surface are known. It is required to determine the temperature at the outer boundary of the cylindrical domain.

We assume that Ω is a two-dimensional convex bounded domain, x^0 is an arbitrary point of Ω, and $\rho < \operatorname{dist}\{x^0, \partial\Omega\}$. We set

$$\Omega_\rho = \{(x_1, x_2) \in \Omega \mid (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 > \rho^2\}.$$

It is clear that $\partial\Omega_\rho = \Gamma_\rho \cup \partial\Omega$, where

$$\Gamma_\rho = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 = \rho^2\}.$$

Consider the process of heating the cylindrical domain

$$D = \Omega_\rho \times \mathbb{R} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in \Omega_\rho, \ -\infty < x_3 < +\infty\}.$$

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The inner cylindrical surface $\Gamma_\rho \times \mathbb{R}$ is the boundary of the heater occupying the region $\{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 < \rho^2,\ x_3 \in \mathbb{R}\}$ and having a time-independent temperature.

Let $u(x_1, x_2, x_3, t)$ be the temperature at a point $(x_1, x_2, x_3) \in D$ at moment $t \geqslant 0$. The heating process is described by the equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\,[k(x)\,\operatorname{grad} u] = 0, \quad x \in D, \quad t > 0, \tag{1.1}$$

where $k(x)$ is the thermal conductivity (see [1]).

On the rod surface, the following conditions are imposed:

$$u(x) = \varphi(x), \quad \frac{\partial u(x)}{\partial r} = \psi(x), \quad x \in \Gamma_\rho, \tag{1.2}$$

where $r = \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2}$.

For known φ and ψ it is required to find the values of $u(x)$ on the outer boundary $\partial\Omega$.

1.2. We are looking for a stationary solution, i.e., a time-independent solution $u = u(x)$. In this case, Equation (1.1) takes the form

$$Lu(x) \;\equiv\; \operatorname{div}\,[k(x)\,\operatorname{grad} u] = 0.$$

We assume that the data of the problem are independent of x_3, i.e., $u = u(x_1, x_2)$.

We introduce the polar coordinates (r, θ) centered at x_0 and assume that the function $k = k(r)$ belongs to $C^1[0, +\infty)$ and satisfies the condition $k(r) > 0$.

The equation of the boundary of Ω is written in the polar coordinates as $r = S(\theta)$, $-pi \leqslant \theta \leqslant \pi$, where $S(\theta)$ is a smooth 2π-periodic function. Then the equation in question has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(rk(r)\frac{\partial u}{\partial r}\Big) + \frac{k(r)}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{1.3}$$

and the boundary conditions (1.2) are written as

$$u(\rho, \theta) = \varphi(\theta), \quad \frac{\partial u(\rho, \theta)}{\partial r} = \psi(\theta), \quad \theta \in [-\pi, \pi], \tag{1.4}$$

where φ and ψ are 2π-periodic functions of θ. In what follows, we assume that the functions φ and ψ belong to the space $L_2[-\pi, \pi]$.

By a *solution to the problem* (1.3), (1.4) we mean a function $u \in C^2(\Omega)$ satisfying Equation (1.3) in Ω_ρ and the boundary conditions (1.2) in the following sense:

$$\lim_{r \to \rho^+} \int_{-\pi}^{\pi} |u(r, \theta) - \varphi(\theta)|^2 \, d\theta = 0, \tag{1.5}$$

$$\lim_{r \to \rho^+} \int_{-\pi}^{\pi} \Big|\frac{\partial u}{\partial r}(r, \theta) - \psi(\theta)\Big|^2 \, d\theta = 0. \tag{1.6}$$

Problem A. For given functions $\varphi \in L_2[-\pi, \pi]$ and $\psi \in L_2[-\pi, \pi]$ find the values of the solution $u(r, \theta)$ to the problem (1.3), (1.4) on $\partial\Omega$.

By a *solution of Problem* A we mean a function $\chi \in L_2(\partial\Omega)$ such that

$$\lim_{\epsilon\to 0}\int\limits_{-\pi}^{\pi} |u(S(\theta)-\epsilon,\theta)-\chi(\theta)|^2\,d\theta = 0. \tag{1.7}$$

The Cauchy problem (1.1), (1.2) was studied in numerous works, starting with the classical work of Hadamard [2] (a review of the corresponding literature can be found in arXiv:0907.2882). As known, the problem (1.1), (1.2) is ill-posed (see [3, 4]). In most works, the stability and regularization are studied (see [3]–[6] and [7]). The computational aspect of this problem was discussed in [7].

The goal of this paper is to obtain easily verifiable conditions guaranteeing the existence of a solution.

1.3. We denote by $E(x,y)$ the fundamental solution to Equation (1.3) which is defined for $|x-x^0|\geqslant\rho$ and $|y-x^0|\geqslant\rho$, decreases at infinity, and satisfies the condition

$$\frac{\partial E(x,x^0+r\theta)}{\partial r}\Big|_{r=\rho} = 0.$$

(see [8, Chapter III, Section 20]). We set

$$P\psi(x) = \int\limits_{-\pi}^{\pi} E(x,x_0+\rho\theta)\psi(\theta)\,d\theta, \quad x\in\mathbb{R}^2.$$

According to the definition of a fundamental solution,

$$LP\psi(x) = 0, \quad x\in\mathbb{R}^2, \quad \frac{\partial P\psi}{\partial r}\Big|_{r=\rho} = \psi(\theta). \tag{1.8}$$

We denote

$$R_1 = \min_{\theta\in\mathbb{T}^1} S(\theta), \quad R_2 = \max_{\theta\in\mathbb{T}^1} S(\theta). \tag{1.9}$$

$$\sigma_1 = \ln\frac{R_1}{\rho}, \quad \sigma_2 = \ln\frac{R_2}{\rho}. \tag{1.10}$$

It is clear that $\sigma_2\geqslant\sigma_1>0$. We note that, in the case $S(\theta)=R=\text{const}$, we have $\sigma_1=\sigma_2=\ln R/\rho$.

We introduce the space A_σ of periodic functions whose extensions are analytic functions. We say that $f\in L_2[-\pi,\pi]$ belongs to the *class* A_σ if

(i) f is a restriction of some holomorfic function $f(\zeta)$ on the stripe $S_\sigma=\{\zeta\in\mathbb{C} : |\operatorname{Im}\zeta|\leqslant\sigma\}$,

(ii) f is a periodic function: $f(\zeta+2\pi)=f(\zeta)$, $\zeta=\xi+i\eta\in S_\sigma$,

(iii) f has the finite norm

$$\|f\|_\sigma^2 = \sup_{|\eta|\leqslant\sigma}\int\limits_{-\pi}^{\pi}|f(\xi+i\eta)|^2\,d\xi<\infty.$$

We formulate the main results of the paper.

Theorem 1.1. *Assume that for given functions φ and ψ there exists a solution to Problem* A. *Then the function*

$$f(\theta) = \varphi(\theta) - P\psi(\rho,\theta) \tag{1.11}$$

belongs to the class A_σ for any $\sigma < \sigma_1$.

Theorem 1.2. *Let the function* (1.11) *belong to the class A_σ for some $\sigma > \sigma_2$. Then the solution to Problem* A *exists and is unique.*

If Ω is a circle centered at point x^0, we prove the following assertion.

Theorem 1.3. *Assume that $S(\theta) = R = \mathrm{const}$. Then a necessary and sufficient condition for the existence of a solution to Problem* A *is that the function* (1.11) *belongs to the class A_σ with $\sigma = \ln R/\rho$.*

The paper is organized as follows. In Section 2, we find a special solution to Equation (1.3) and evaluate this solution together with its derivatives. In Section 3, we obtain sufficient conditions for the existence of a solution to the problem (1.3), (1.4). In Section 4, we prove Theorems 1.1–1.3.

2 Preliminary Estimates

We begin by considering the problem (1.3), (1.4) with $\psi = 0$. In this case, the condition (1.4) takes the form

$$u(\rho,\theta) = \varphi(\theta), \quad \frac{\partial u(\rho,\theta)}{\partial r} = 0, \quad \theta \in [-\pi,\pi]. \tag{2.1}$$

First of all, we note that the problem (1.3), (1.4) significantly differs from the problem (1.3), (2.1) by the conditions imposed on the function φ. Namely, for any smooth function φ it is always possible to choose a function ψ such that the problem (1.3), (1.4) has a solution (see [9] for the Dirichlet–Neumann operator). But the situation is much more complicated for the problem (1.3), (2.1). In this paper, we show that a necessary condition for the solvability of the problem (1.3), (2.1) is that the function φ belongs to some class of analytic functions. For the existence of a solution, it is enough that the function φ belongs to a similar, but narrower class. The solution to Problem A depends on the distance between the point x_0 and the boundary $\partial\Omega$ of Ω.

We find a solution to the Cauchy problem (1.3), (2.1) in a disk $D(x_0, R)$ of an arbitrary radius $R > \rho$ centered at a point x^0. For this purpose we first assume that the solution to the problem (1.3), (2.1) has the form $u(r,\theta) = v(r)e^{im\theta}$. Then Equation (1.3) can be written as

$$\frac{1}{rk(r)}\frac{\partial}{\partial r}\Big(rk(r)\frac{\partial v}{\partial r}\Big) - \frac{m^2}{r^2}v(r) = 0, \quad \rho < r < R. \tag{2.2}$$

Taking into account (2.1), we consider the boundary conditions

$$v(\rho) = 1, \quad v'(\rho) = 0. \tag{2.3}$$

We denote by $V_m(r)$ the solution to the boundary value problem (2.2), (2.3) (for the existence of a solution, see [10, Chapter 3, Section 6]). By (2.2) and the second condition in (2.3), the function $V_m(r)$ satisfies the equation

$$V_m'(r) = \frac{m^2}{rk(r)} \int\limits_\rho^r \frac{k(s)}{s} V_m(s)\, ds, \quad \rho \leqslant r \leqslant R. \tag{2.4}$$

It is clear that $V_0(r) \equiv 1$.

Further, in the case $m \geqslant 1$, from the positivity of $k(r)$ and the first condition in (2.3) it follows that the derivative $V_m'(r)$ is positive on the interval $\rho < r \leqslant R$:

$$V_m'(r) \ > 0, \quad \rho < r \leqslant R, \quad m \geqslant 1. \tag{2.5}$$

According to the first condition in (2.3),

$$V_m(r) \ \geqslant \ 1, \quad \rho \leqslant r \leqslant R, \quad m \geqslant 0. \tag{2.6}$$

Lemma 2.1. *The solution $V_m(r)$ to the boundary problem* (2.2), (2.3) *with some positive constants $C_j = C_j(\rho, R)$ satisfies the estimate*

$$C_1 \Big(\frac{r}{\rho}\Big)^m \leqslant V_m(r) \ \leqslant C_2 \Big(\frac{r}{\rho}\Big)^m, \quad \rho \leqslant r \leqslant R. \tag{2.7}$$

Proof. Recall that k is a continuously differentiable function such that $k(r) > 0$. We can write Equation (2.2) as

$$r^2 v''(r) + \Big[1 + r\frac{k'(r)}{k(r)}\Big] r v'(r) \ - \ m^2 v(r) = 0. \tag{2.8}$$

We set $r = \rho e^t$ and $w(t) = v(r) = v(\rho e^t)$. Then $rv'(r) = w'(t)$ and $r^2 v''(r) = w''(t) - w'(t)$. Equation (2.2) takes the form

$$w''(t) + \frac{k'(e^t)}{k(e^t)} e^t w'(t) - m^2 w(t) = 0, \quad 0 < t < \ln\frac{R}{\rho}. \tag{2.9}$$

It is known [11, Chapter II, Section 4, Theorem 1] that Equation (2.9) has two solutions

$$w_1(t) = e^{mt}\Big[1 + \frac{O(1)}{m}\Big], \quad w_2(t) = e^{-mt}\Big[1 + \frac{O(1)}{m}\Big], \tag{2.10}$$

$$w_1'(t) = me^{mt}\Big[1 + \frac{O(1)}{m}\Big], \quad w_2'(t) = -me^{-mt}\Big[1 + \frac{O(1)}{m}\Big]. \tag{2.11}$$

We note that the function $w(t) = w_1'(0)w_2(t) - w_2'(0)w_1(t)$ is a solution to Equation (2.9) such that $w'(0) = 0$. By the uniqueness of a solution to the Cauchy problem, we have

$$w(0) = w_1'(0)w_2(0) - w_2'(0)w_1(0) \neq 0.$$

We set $h = \ln R/\rho$ and consider the function

$$W(t) = \frac{w_1'(0)w_2(t) - w_2'(0)w_1(t)}{w_1'(0)w_2(0) - w_2'(0)w_1(0)}, \quad 0 \leqslant t \leqslant h. \tag{2.12}$$

It is clear that $W(t)$ satisfies Equation (2.9) and the boundary conditions $W(0)=1$, $W'(0)=0$. By (2.10) and (2.11),

$$w_1'(0)w_2(t) - w_2'(0)w_1(t) = m\Big[1+\frac{O(1)}{m}\Big]e^{-mt}\Big[1+\frac{O(1)}{m}\Big] + m\Big[1+\frac{O(1)}{m}\Big]e^{mt}\Big[1+\frac{O(1)}{m}\Big]$$

$$= me^{-mt}\Big[1+\frac{O(1)}{m}\Big] + me^{mt}\Big[1+\frac{O(1)}{m}\Big].$$

Hence

$$w_1'(0)w_2(t) - w_2'(0)w_1(t) = 2m\cosh mt\Big[1+\frac{O(1)}{m}\Big], \quad t>0.$$

Further,

$$w_1'(0)w_2(0) - w_2'(0)w_1(0) = 2m\Big[1+\frac{O(1)}{m}\Big].$$

Therefore, we obtain the inequality

$$|w_1'(0)w_2(0) - w_2'(0)w_1(0)| \geqslant Cm, \quad m\in\mathbb{N},$$

with some constant $C>0$. Consequently, for the function (2.12) the following asymptotic equality holds:

$$W(t) = \cosh mt\Big[1+\frac{O(1)}{m}\Big], \quad t>0,$$

which implies that for sufficiently large m

$$C_1\cosh mt \leqslant W(t) \leqslant C_2\cosh mt.$$

Therefore, for sufficiently large m

$$C_1 e^{mt} \leqslant W(t) \leqslant C_2 e^{mt}.$$

Taking into account that $r=\rho e^t$, we conclude that for $m\geqslant N$

$$C_1\Big(\frac{r}{\rho}\Big)^m \leqslant W(t) \leqslant C_2\Big(\frac{r}{\rho}\Big)^m, \quad r\geqslant\rho, \tag{2.13}$$

where N is large enough. We note that the right inequality in (2.13) holds for any $m\in\mathbb{N}$.

Now, we set

$$V_m(r) = W\Big(\ln\frac{r}{\rho}\Big) = W(t).$$

Then (2.13) with $m\geqslant N$ implies the required estimate (2.7).

In the case $m<N$, we can use the estimate (2.6). Setting $C_1=(\rho/R)^N$, we complete the proof of Lemma 2.1. □

Lemma 2.2. *The derivatives of the solution $V_m(r)$ to the boundary problem* (2.2), (2.3) *satisfy the estimates*

$$|V_m'(r)| \leqslant Cm\Big(\frac{r}{\rho}\Big)^m, \quad |V_m''(r)| \leqslant Cm^2\Big(\frac{r}{\rho}\Big)^m, \quad r\geqslant\rho. \tag{2.14}$$

Proof. From (2.4)–(2.7) we obtain the first estimate in (2.14):

$$V_m'(r) = O(m^2)\int_\rho^r V_m(s)\,ds = O(m^2)\int_\rho^r \Big(\frac{s}{\rho}\Big)^m ds = O(m)\Big(\frac{r}{\rho}\Big)^m.$$

The second estimate in (2.14) immediately follows from Equation (2.8), the inequality (2.7), and the first estimate in (2.14). □

3 Existence of Solution to Auxiliary Problem

We show that the solution to the boundary value problem (1.3), (2.1) can be represented as the series

$$u(r,\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta) V_m(r), \tag{3.1}$$

where a_m and b_m are the Fourier coefficients of φ, i.e.,

$$\varphi(\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta). \tag{3.2}$$

The proof relies on the following property of the class A_σ.

Lemma 3.1. *A function $\varphi \in L_2(\mathbb{T})$ belongs to the class A_σ if and only if*

$$\sum_{m=1}^{\infty} (a_m^2 + b_m^2) e^{2\sigma m} < \infty. \tag{3.3}$$

In this case,

$$\frac{1}{2\pi} \|\varphi\|_\sigma^2 \leqslant \frac{a_0^2}{2} + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) e^{2\sigma m} \leqslant \frac{2}{\pi} \|\varphi\|_\sigma^2. \tag{3.4}$$

Proof. We write the Fourier series (3.2) in the complex form

$$\varphi(\theta) = \sum_{m=-\infty}^{\infty} c_m e^{im\theta},$$

where

$$c_0 = \frac{a_0}{2}, \quad c_m = \frac{a_m + i b_m}{2}, \quad c_{-m} = \frac{a_m - i b_m}{2}, \quad m \geqslant 1. \tag{3.5}$$

Next, we use the fact that the function $\varphi(\theta)$ in the class A_σ satisfies the inequalities

$$\frac{1}{4\pi} \|\varphi\|_\sigma^2 \leqslant \sum_{m=-\infty}^{\infty} |c_m|^2 \cosh 2\sigma m \leqslant \frac{1}{2\pi} \|\varphi\|_\sigma^2. \tag{3.6}$$

We note that the constants in these inequalities are exact (see [12]).

Further, taking into account (3.5), we get

$$\begin{aligned} \sum_{m=-\infty}^{\infty} |c_m|^2 \cosh 2\sigma m &= |c_0|^2 + \sum_{m=1}^{\infty} (|c_{-m}|^2 + |c_m|^2) \cosh 2\sigma m \\ &= \frac{a_0^2}{4} + \frac{1}{2} \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \cosh 2\sigma m. \end{aligned}$$

Substituting this value into the inequality (3.6), we get

$$\frac{1}{2\pi} \|\varphi\|_\sigma^2 \leqslant \frac{a_0^2}{2} + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \cosh 2\sigma m \leqslant \frac{1}{\pi} \|\varphi\|_\sigma^2 \tag{3.7}$$

The required estimate (3.3) follows from (3.7) and the obvious inequalities

$$\frac{e^{2\sigma m}}{2} \leqslant \cosh 2\sigma m \leqslant e^{2\sigma m}.$$

Suppose that the series (3.3) converges. For an arbitrary $N \in \mathbb{N}$ we consider the function

$$\varphi_N(\theta) = \frac{a_0}{2} + \sum_{m=1}^{N} (a_m \cos m\theta + b_m \sin m\theta).$$

The sequence of φ_N converges to the function φ in the $L_2(\mathbb{T})$-metric as $N \to \infty$. In addition, according to the left inequality in (3.4), this sequence is bounded in the A_σ-metric. Therefore, the limit function φ belongs to the space A_σ. □

Lemma 3.2. *Let φ belong to the class A_σ with $\sigma = \ln R/\rho$. Then for some positive constants C_j, $j = 1, 2$,*

$$C_1\|\varphi\|_\sigma^2 \leqslant \frac{a_0^2}{2} + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) V_m^2(R) \leqslant C_2\|\varphi\|_\sigma^2. \tag{3.8}$$

Proof. According to Lemma 2.1, the inequalities (3.8) are equivalent to the inequalities

$$C_1'\|\varphi\|_\sigma^2 \leqslant \frac{a_0^2}{2} + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \Big(\frac{r}{\rho}\Big)^{2m} \leqslant \ C_2'\|\varphi\|_\sigma^2.$$

Since $R/\rho = e^\sigma$, we can write

$$C_1'\|\varphi\|_\sigma^2 \leqslant \frac{a_0^2}{2} + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) e^{2\sigma m} \leqslant \ C_2'\|\varphi\|_\sigma^2.$$

The last inequalities follow from Lemma 3.1. □

Lemma 3.3. *Let φ belong to the class A_σ with $\sigma = \ln R/\rho$. Then the function $u(r, \theta)$ defined by (3.1) belongs to $C^2([\rho, R) \times [-\pi, \pi])$.*

Proof. Indeed, for $k = 1, 2$ we set

$$u^{(k,2)}(r, \theta) = \Big(\frac{\partial}{\partial r}\Big)^k \Big(\frac{\partial}{\partial \theta}\Big)^2 u(r, \theta).$$

According to (3.1), we can write

$$u^{(k,2)}(r, \theta) = -\sum_{m=1}^{\infty} m^2 (a_m \cos m\theta + b_m \sin m\theta) V_m^{(k)}(r). \tag{3.9}$$

We set

$$d_m = (a_m^2 + b_m^2) \Big(\frac{R}{\rho}\Big)^{2m}.$$

By Lemmas 2.1 and 3.2, the following series is convergent:

$$\sum_{m=1}^{\infty} d_m \ \leqslant C \sum_{m=1}^{\infty} (a_m^2 + b_m^2) V_m^2(R) \ \leqslant C\|\varphi\|_\sigma^2.$$

In this case, from (3.9) and Lemma 2.2 we get

$$|u^{(k,2)}(r,\theta)|^2 \leqslant \Big(\sum_{m=1}^{\infty} |a_m \cos m\theta + b_m \sin m\theta|^2 |V_m^{(k)}(r)|^2 m^6 \Big) \sum_{m=1}^{\infty} \frac{1}{m^2}$$

$$\leqslant C \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \Big(\frac{r}{\rho}\Big)^{2m} m^{2k+6} = C \sum_{m=1}^{\infty} d_m \Big(\frac{r}{R}\Big)^{2m} m^{2k+6}. \tag{3.10}$$

For any r in the interval $0 < r < R$ we introduce the quantity

$$M(r/R) = \max_{m\in\mathbb{N}} \Big(\frac{r}{R}\Big)^{2m} m^{10}.$$

Substituting this quantity into (3.10), we obtain the required estimate

$$|u^{(k,2)}(r,\theta)|^2 \leqslant CM(r/R) \sum_{m=1}^{\infty} d_m \ \leqslant CM(r/R)\|\varphi\|_\sigma^2.$$

The lemma is proved. □

Lemma 3.4. *Let φ belong to the class A_σ with $\sigma = \ln R/\rho$. Then for each $r \in [\rho, R]$ the function $u(r,\theta)$ defined by* (3.1) *belongs to $L_2[-\pi,\pi]$ and*

$$\lim_{r\to R} \int_{-\pi}^{\pi} \big|u(r,\theta) - u(R,\theta)\big|^2 \, d\theta = 0. \tag{3.11}$$

Proof. Indeed, according to the Parseval theorem and Lemma 3.2, we have

$$\|u(r,\cdot)\|^2 = \frac{\pi}{2} a_0^2 + \pi \sum_{m=1}^{\infty} (a_m^2 + b_m^2) V_m^2(r) \leqslant C\|\varphi\|_\sigma^2, \quad \rho \leqslant r \leqslant R.$$

Taking into account (2.5), we get

$$\int_{-\pi}^{\pi} |u(R,\theta) - u(r,\theta)|^2 \, d\theta = \pi \sum_{m=1}^{\infty} (a_m^2 + b_m^2)[V_m(R) - V_m(r)]^2$$

$$= \pi \sum_{m=1}^{\infty} (a_m^2 + b_m^2) V_m^2(R) \Big[1 - \frac{V_m(r)}{V_m(R)}\Big]^2 \leqslant \pi \sum_{m=1}^{\infty} (a_m^2 + b_m^2) V_m^2(R) \leqslant C\|\varphi\|_\sigma^2.$$

We note that each term is a continuous function in $r \in [\rho, R]$ and the series is majorized by a numerical series. Therefore, according to the Weierstrass theorem (see [13, Theorem 7.10]), the series converges uniformly with respect to $r \in [\rho, R]$ and is a continuous function of r. Hence the equality (3.11) follows. □

Lemma 3.5. *Let φ belong to the class A_σ with $\sigma = \ln R/\rho$. Then the function $u(r,\theta)$ defined by* (3.1) *is a solution to Problem* A *in the domain*

$$\Omega_R = \{x \in \mathbb{R}^2 \ : \ \rho < |x - x^0| < R\} \tag{3.12}$$

with boundary conditions (2.1).

Proof. According to Lemmas 3.3 and 3.4, the function $u(r,\theta)$ is twice continuously differentiable in Ω_R and satisfies Equation (1.3) with boundary conditions (2.1). We set $\chi(\theta) = u(R,\theta)$. Then (1.7) follows from Lemma 3.4. $\square$

Lemma 3.6. *Let $u(r,\theta)$ be a solution to Problem* A *in the domain* (3.12) *with boundary conditions* (2.1). *Then φ belongs to the class A_σ with $\sigma = \ln R/\rho$.*

Proof. We expand the solution $u(r,\theta)$ into the Fourier series

$$u(r,\theta) = \sum_{m\in\mathbb{Z}} c_m(r)e^{im\theta}, \quad \rho \leqslant r \leqslant R,$$

where

$$c_m(r) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} u(r,\theta)e^{-im\theta}d\theta, \quad \rho \leqslant r \leqslant R. \tag{3.13}$$

By the definition of a solution, $u(r,\theta)$ is twice continuously differentiable in the annulus $(\rho,R)\times[-\pi,\pi]$ and satisfies Equation (1.3). In this case, from (3.13) it follows that the coefficients $c_m(r)$ satisfy (2.2) in the interval $\rho < r < R$. Indeed,

$$\frac{1}{rk(r)}\frac{\partial}{\partial r}\Big(rk(r)\frac{\partial c_m(r)}{\partial r}\Big) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\frac{1}{rk(r)}\frac{\partial}{\partial r}\Big(rk(r)\frac{\partial u(r,\theta)}{\partial r}\Big)e^{-im\theta}d\theta$$

$$= -\frac{1}{\pi r^2}\int\limits_{-\pi}^{\pi}\frac{\partial^2 u(r,\theta)}{\partial\theta^2}e^{-im\theta}d\theta = \frac{m^2}{\pi r^2}\int\limits_{-\pi}^{\pi}u(r,\theta)e^{-im\theta}d\theta = \frac{1}{m^2}\,c_m(r).$$

Denote by $U_m(r)$ the solution to Equation (2.2) satisfying the boundary conditions

$$U_m(0) = 0, \quad U_m'(0) = 1. \tag{3.14}$$

Then

$$c_m(r) = A_m V_m(r) + B_m U_m(r), \quad \rho < r < R.$$

Therefore, the solution takes the form

$$u(r,\theta) = \sum_{m\in\mathbb{Z}}[A_m V_m(r) + B_m U_m(r)]e^{im\theta}, \quad \rho < r < R.$$

Hence

$$u_r'(r,\theta) = \sum_{m\in\mathbb{Z}}[A_m V_m'(r) + B_m U_m'(r)]e^{im\theta}, \quad \rho < r < R.$$

According to the conditions (1.6) and (2.1), the derivative $u_r'(r,\theta)$ converges to 0 strongly in the $L_2(\mathbb{T})$-metric as $r\to\rho$. Therefore, it converges weakly. Then, due to (2.1), we get

$$A_m V_m'(\rho) + B_m U_m'(\rho) = \lim_{r\to\rho^+}\,[A_m V_m'(r) + B_m U_m'(r)] = \lim_{r\to\rho^+}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}u'(r,\theta)e^{-im\theta}d\theta = 0.$$

Taking into account (2.3) and (3.14), we have $B_m = 0$.

Analogously, by (1.5) and the condition $V_m(\rho) = 1$,

$$A_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(\theta) e^{-im\theta} d\theta.$$

Therefore, the solution should have the form (3.1).

The condition (1.7) means that there exists a function $\chi(\theta) \in L_2(\mathbb{T})$ such that $u(r, \theta)$ converges to $\chi(\theta)$ as $r \to R$. Hence, according to (3.1),

$$\frac{\pi}{2} a_0^2 + \pi \sum_{m=1}^{\infty} (a_m^2 + b_m^2) V_m^2(R) = \lim_{r \to R} \|u(r, \cdot)\|^2 = \|\psi\|^2 < \infty.$$

In this case, the fact that φ belongs to the class A_σ follows from Lemma 3.2. □

Remark 3.1. A solution to the problem (1.3), (2.1) is unique because the Fourier expansion of any solution $u(r, \theta)$ has the form (3.1), (3.2). Therefore, the problem (1.3), (1.4) can have only one solution.

4 Proof of Theorems 1.1–1.3

We begin by proving Theorem 1.3. The proof is based on the results of Section 3.

Proof of Theorem 1.3. *Necessity.* Let $u(r, \theta)$ be the solution to Problem A in the case $S(\theta) = R = \text{const}$. Consider the function

$$v(r, \theta) = u(r, \theta) \ - \ P\psi(r, \theta). \tag{4.1}$$

By (1.8), the function $v(r, \theta)$ satisfies the equation

$$Lv(r, \theta) \equiv \frac{1}{r} \frac{\partial}{\partial r} \Big(r k(r) \frac{\partial v}{\partial r} \Big) + \frac{k(r)}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0, \quad \rho < r < R, \tag{4.2}$$

and boundary conditions

$$v(\rho, \theta) = f(\theta), \quad \frac{\partial v(\rho, \theta)}{\partial r} = 0, \quad \theta \in [-\pi, \pi], \tag{4.3}$$

where $f(\theta)$ is defined by (1.11).

It is clear that the function $v(r, \theta)$ is a solution to Problem A with boundary conditions (4.3). Then, according to Lemma 3.6, $f(\theta)$ belongs to the class A_σ with $\sigma = \ln R/\rho$.

Sufficiency. Let the function (1.11) belong to the class A_σ with $\sigma = \ln R/\rho$. According to Lemma 3.5, the solution to the problem (4.2), (4.3) exists. It is clear that the function $u(r, \theta) = v(r, \theta) + P\psi(r, \theta)$ is a solution to Problem A with boundary conditions (1.4). □

Proof of Theorem 1.2. Let the function (1.11) belong to the class A_σ for some $\sigma > \sigma_2$, where σ_2 is defined by (1.9) and (1.10). We set $R = \rho e^\sigma$ and denote by $u(r, \theta)$ a solution (existing by Theorem 1.3) to Problem A in the annulus $(\rho, R) \times [-\pi, \pi]$. We set $\chi(\theta) = u(S(\theta), \theta)$. Then

$$\chi(\theta) - u(S(\theta) - \epsilon, \theta) = u(S(\theta), \theta) - u(S(\theta) - \epsilon, \theta) = \int_0^{\epsilon} \frac{\partial u}{\partial r}(S(\theta) - t, \theta)\, dt.$$

Further,

$$|\chi(\theta) - u(S(\theta) - \epsilon, \theta)|^2 \leqslant \epsilon \int\limits_0^\epsilon \Big|\frac{\partial u}{\partial r}(S(\theta) - t, \theta)\Big|^2 \, dt \leqslant \epsilon \int\limits_\rho^{S(\theta)} \Big|\frac{\partial u}{\partial r}(S(r, \theta)\Big|^2 \, dr.$$

Hence

$$\int\limits_{-\pi}^{\pi} |\chi(\theta) - u(S(\theta) - \epsilon, \theta)|^2 \, d\theta \leqslant \epsilon \int\limits_{\Omega_\rho} \Big|\frac{\partial u}{\partial r}(S(r, \theta)\Big|^2 \, dr \, d\theta.$$

By Lemma 3.3, the condition (1.7) is satisfied. Consequently, the function $u(r, \theta)$ is a solution to Problem A in Ω_ρ.

The uniqueness of the solution follows from Remark 3.1. □

Proof of Theorem 1.1. Let $u(r, \theta)$ be a solution to Problem A in Ω_ρ. We show that the function (1.11) belongs to the class A_σ for any $\sigma < \sigma_1$, where σ_1 is defined by (1.9) and (1.10). We set $R = \rho e^\sigma$. It is clear that the function (4.1) is a solution to the boundary value problem (4.2), (4.3) in the domain Ω_ρ. Moreover, the function (4.1) is also a solution to the problem (4.2), (4.3) in the smaller domain $(\rho, R) \times [-\pi, \pi]$. According to Lemma 3.6, we have $f \in A_\sigma$. □

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